5.

Quadratic functions

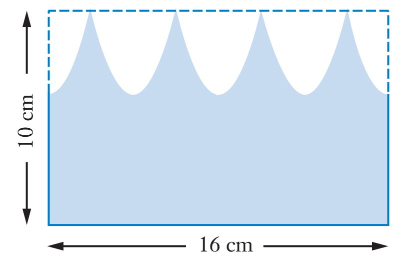
- Quadratic functions
- Key features of the graph of $y = a(x - p)^2 + q$
- Determining the rule from key features of the graph
- Determining the key features of the graph from the rule
- Tables of values
- Completing the square
- Miscellaneous exercise five

The *Preliminary work* reminded you of what the graph of $y = x^2$ looks like but what of the graphs of $y = x^2 + 3$ or $y = (x - 3)^2$ or $y = (x - 3)^2 + 4$, etc. Are you familiar with the effects changing a, p and q have on the graph of $y = a(x - p)^2 + q$?

If you think you can remember something about these things then continue on to the 'situation' described below. If you are not familiar with any of this work on the graphs of quadratic functions then read the next page and work through **Exercise 5A** first and then come back to the situation described below.

Situation

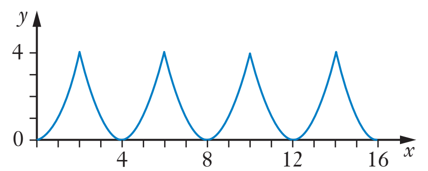
An engineer needs to write instructions for a computer operated cutting machine.



The machine cuts four 'teeth' along one of the long sides of a rectangular piece of metal, 16 cm × 10 cm.

The instructions need to give the height of the teeth and the equations of the cuts required.

The particular job shown requires the teeth to be cut as shown on the right.



The engineer starts writing the instructions as follows:

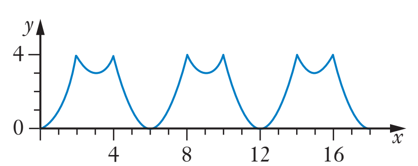
'Height' of teeth 4 cm. For $0 \le x < 2$, cut $y = x^2$.

For the next instruction, i.e. for $2 \le x < 6$, the engineer is not sure which of the following equations to use:

$y = x^2 + 4$, $\quad y = x^2 - 4$, $\quad y = (x - 4)^2$, $\quad y = (x + 4)^2$.

Decide which it should be and then write out the complete instructions for $0 \le x \le 16$.

The machine is now required to cut teeth as shown on the right, in a sheet of metal that is 18 cm long.



The engineer began her instructions:

'Height' of teeth 4 cm. For $0 \le x < 2$, cut $y = x^2$.

Again the engineer is not sure which equation to use for the next instruction, i.e. for

$$2 \le x < 4.$$

She knows that it should be one of the following:

$y = (x - 3)^2 + 3$, $\quad y = (x - 3)^2 - 3$, $\quad y = (x + 3)^2 + 3$, $\quad y = (x + 3)^2 - 3$.

Decide which it should be and then write out the complete instructions for $0 \le x \le 18$.

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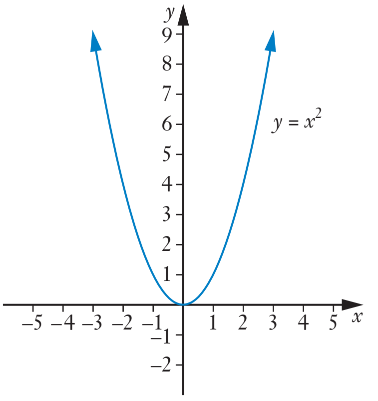
Quadratic functions

The points indicated in the table below would clearly *not* lie in a straight line because as the x value increases by 1 the y value does *not* alter by a constant amount.

x	–3	–2	–1	0	1	2	3
y	9	4	1	0	1	4	9

As you probably recognised, the table is for $y = x^2$, the most basic quadratic function.

The graph of $y = x^2$ is shown on the right and is said to be **parabolic** in shape. The curve is a known as a **parabola**.



Note that:

- the graph is symmetrical with the y-axis as the line of symmetry.
- the graph is shaped like 'a valley' coming down to a 'low point' called the **minimum** point.
- for larger and larger positive x values, y takes even larger positive values. We say that 'as x approaches infinity then y approaches infinity' (and indeed it does it faster than x does). This is written: As $x \to \infty$ then $y \to \infty$.
- Similarly: As $x \to -\infty$ then $y \to \infty$.

All functions that can be written in the form

$$y = ax^2 + bx + c \quad \text{for } a \neq 0$$

will have graphs that are the same basic shape as that of $y = x^2$ but may be moved left, right, up, down, flipped over, squeezed or stretched.

Just as the equations of linear functions are not always presented in the form

$$y = mx + c$$

(for example $2y = 3x + 4$, $3x + 4y = 7$, $6x + y - 5 = 0$), then so the equations of quadratic functions are not always presented in the form

$$y = ax^2 + bx + c.$$

Indeed for quadratic functions there are three ways in which the rule is frequently given:

	$y = ax^2 + bx + c,$	for example	$y = 2x^2 - 12x + 10,$
	$y = a(x - b)(x - c),$	for example	$y = 2(x - 1)(x - 5),$
and	$y = a(x - b)^2 + c,$	for example	$y = 2(x - 3)^2 - 8.$

The reader should confirm that by expanding the brackets and collecting like terms the last two forms can be written in the form $y = ax^2 + bx + c$, and indeed are just alternative ways of writing the first equation, $y = 2x^2 - 12x + 10$.

Whilst the letters a, b and c are quite commonly used this does not have to be the case.
Exercise 5A which follows, considers quadratic functions with equations given in the form

$$y = a(x - p)^2 + q$$

and investigates how changing the values of a, p and q alters the graph of the function.

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Exercise 5A

This exercise should refresh your memory as to the effect altering the values a, p and q have on the graph of $y = a(x - p)^2 + q$.

Either work through the exercise as given below or, alternatively, use the ability of some calculators or internet sites to automatically move through displays of $y = ax^2$, $y = x^2 + q$ or $y = (x - p)^2$ for changing values of a, p and q to explore these aspects.

1 Changing the value of 'a' in $y = ax^2$.

Display the graphs of the following functions altogether on the screen of a graphic calculator using an x-axis from –4 to 4 and a y-axis from –10 to 10.

$$y = x^2, \quad y = 2x^2, \quad y = 4x^2, \quad y = 0.5x^2, \quad y = -2x^2.$$

Write a few sentences explaining the effect changing the value of a has on the graph of $y = ax^2$, referencing your comments to how the graphs differ from that of $y = x^2$.

2 Changing the value of 'q' in $y = x^2 + q$.

Display the graphs of the following functions altogether on the screen of a graphic calculator using an x-axis from –3 to 3 and a y-axis from –4 to 14.

$$y = x^2, \quad y = x^2 + 2, \quad y = x^2 + 4, \quad y = x^2 - 3.$$

Write a few sentences explaining the effect changing the value of q has on the graph of $y = x^2 + q$. Reference your comments to how the various graphs differ from that of $y = x^2$.

3 Changing the value of 'p' in $y = (x - p)^2$.

Display the graphs of the following functions altogether on the screen of a graphic calculator using an x-axis from –6 to 8 and a y-axis from –10 to 10.

$$y = x^2, \quad y = (x - 2)^2, \quad y = (x - 5)^2, \quad y = (x + 3)^2.$$

Write a few sentences explaining the effect changing the value of p has on the graph of $y = (x - p)^2$. Reference your comments to how the various graphs differ from that of $y = x^2$.

4 Display the graphs of the two functions below on the screen of a graphic calculator using an x-axis from –6 to 6 and a y-axis from –8 to 16.

$$y = (x - 3)^2 + 4, \quad y = 2(x + 1)^2 - 4$$

Check that the location of the graphs on the calculator display agrees with your expectations as a result of doing questions 1, 2 and 3.

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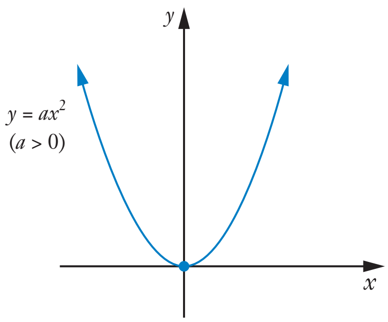
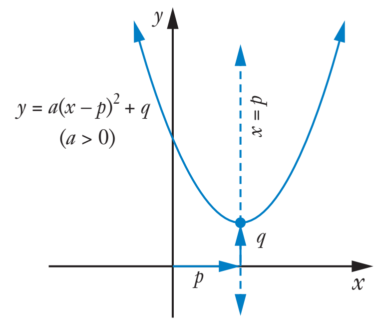
Key features of the graph of $y = a(x - p)^2 + q$

You should have discovered that the graph of $y = a(x - p)^2 + q$ has the same shape as that of $y = ax^2$ but has moved p units to the right and q units up.

Thus:

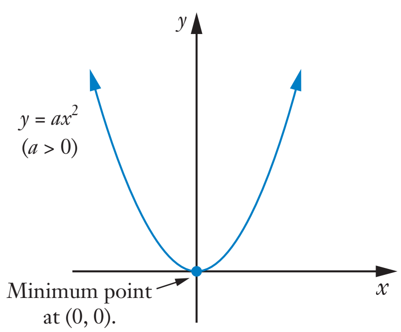
- whilst the graph of $y = ax^2$ has the y-axis as its line of symmetry, the graph of $y = a(x - p)^2 + q$ has the line $x = p$ as its line of symmetry.

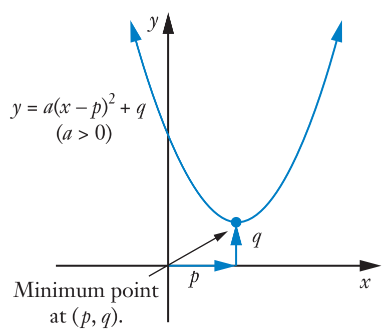




- whilst the graph of $y = ax^2$ has a turning point at (0, 0), the graph of $y = a(x - p)^2 + q$ has a turning point at (p, q).

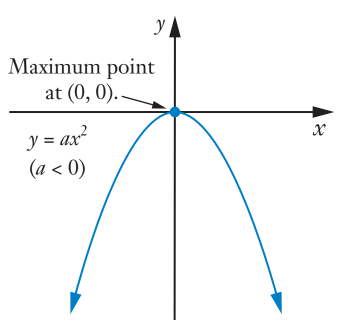
  For $a > 0$ this will be a **minimum** turning point:

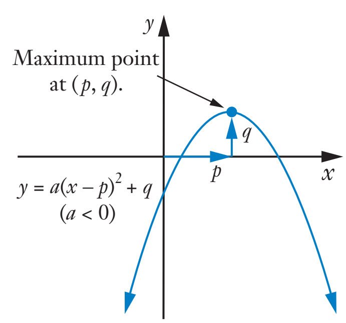




  Note: This 'valley shape' is sometimes referred to as being **concave up**.

  For $a < 0$ this will be a **maximum** turning point:





  Note: This 'hill shape' is sometimes referred to as being **concave down**.

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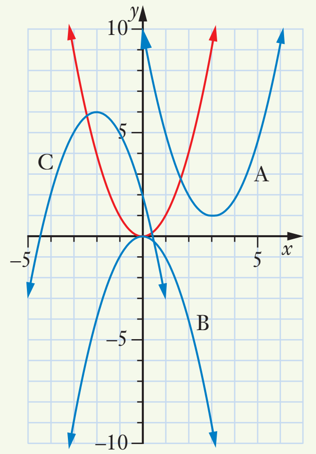
Determining the rule from key features of the graph

EXAMPLE 1

In the graph shown, the curves A, B and C are identical in shape to the curve shown in red, though B and C are 'upside down' versions. The red curve has equation $y = x^2$. Write down the equations of curves A, B and C.



Solution

Curve A is the red curve, $y = x^2$, moved right 3 and up 1.

Curve A has equation $y = (x - 3)^2 + 1$.

Curve B is the red curve, $y = x^2$, reflected in the x axis.

Curve B has equation $y = -x^2$.

Curve C is curve B, $y = -x^2$, moved 2 units left and 6 units up.

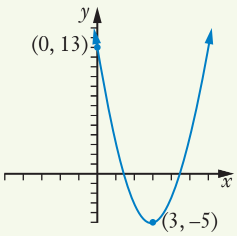
Thus curve C has equation $y = -1(x + 2)^2 + 6$.

I.e. $y = -(x + 2)^2 + 6$.

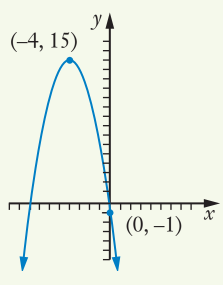
EXAMPLE 2

Given that each of the following graphs show quadratic functions, determine the rule for each function.

a



b



Solution

a Given that the graph shows a quadratic function with a turning point at (3, –5) the equation will be of the form $y = a(x - 3)^2 - 5$.

(0, 13) must 'fit' the equation, hence $13 = a(0 - 3)^2 - 5$,

so $13 = 9a - 5$.

Solving gives $a = 2$.

The given graph has equation $y = 2(x - 3)^2 - 5$.

b Given that the graph shows a quadratic function with a turning point at (–4, 15) the equation will be of the form $y = a(x + 4)^2 + 15$.

(And because of the *maximum* turning point we expect a to be negative.)

(0, –1) must 'fit' the equation, hence $-1 = a(0 + 4)^2 + 15$,

so $-16 = 16a$.

Hence $a = -1$.

The given graph has equation $y = -(x + 4)^2 + 15$.

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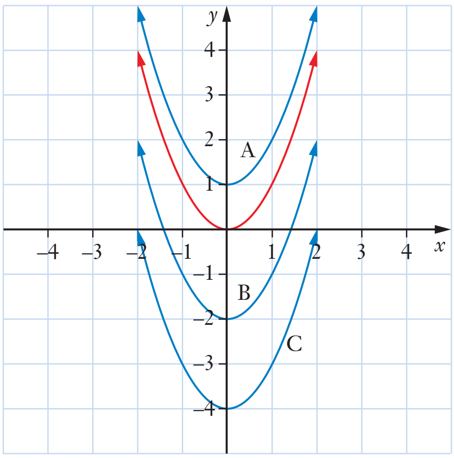
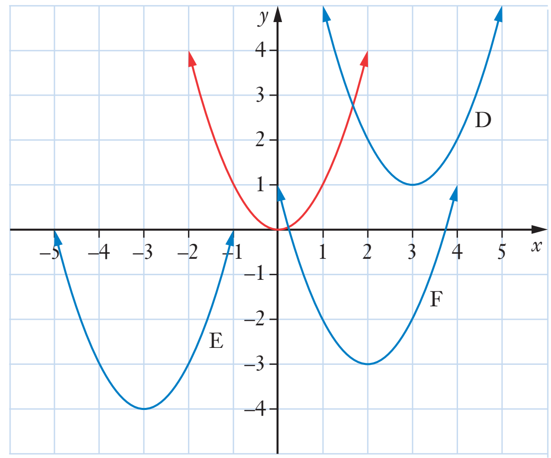
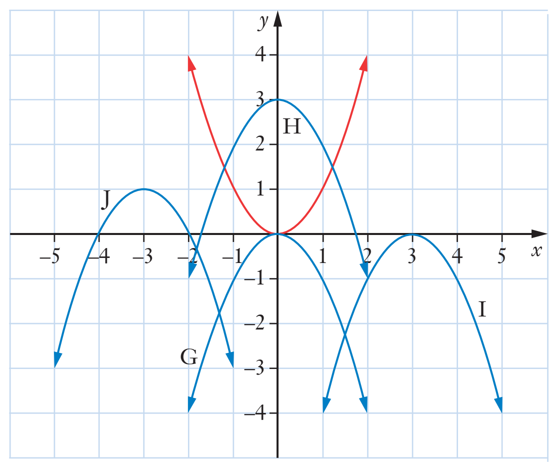
Exercise 5B

1 In the graphs below, the curves A, B, C, D, E and F are identical in shape to the red curve shown, which has equation $y = x^2$. Write down the equations of curves A, B, C, D, E and F.

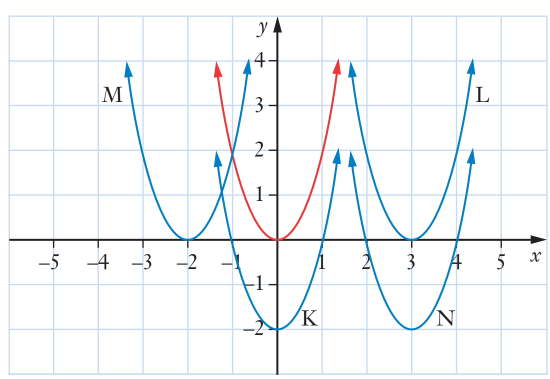




2 The curves G, H, I and J are all identical in shape to the red curve shown (but are each 'upside down' versions). If the red curve has equation $y = x^2$, write down the equations of curves G, H, I and J.



3 The curves K, L, M and N are all of the same shape as the red curve shown. If the red curve has equation $y = 2x^2$, write down the equations of curves K, L, M and N.



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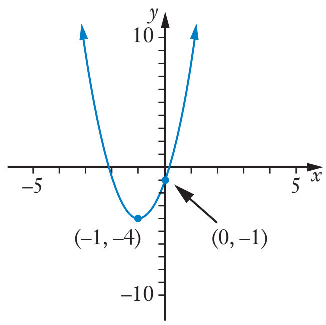
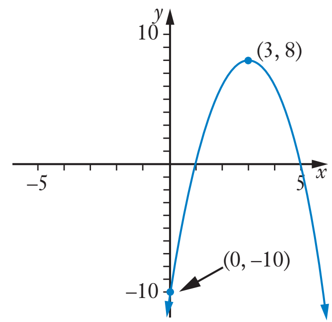
4 Given that each of the following graphs show quadratic functions, determine the equation of each one.

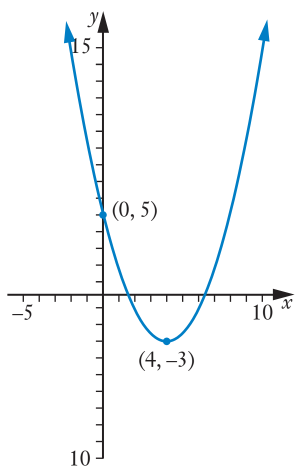
a



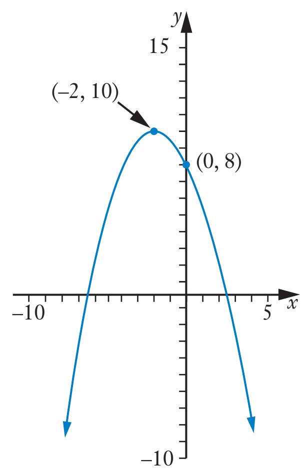
b



c



d



Determining the key features of the graph from the rule

Each of the three ways the rules for quadratic functions are usually given, i.e.:

$$y = ax^2 + bx + c, \qquad y = a(x - b)(x - c), \qquad y = a(x - b)^2 + c$$

readily provide information about some of the key features of the graph of the function.

Note: In the expression $ax^2 + bx + c$, a is said to be the **coefficient** of x^2, and b is said to be the **coefficient** of x.

Rule given in the form $y = a(x - b)^2 + c$

- *Line of symmetry*

  The graph of $y = a(x - b)^2 + c$ has the line $x = b$ as its line of symmetry.

- *Turning point*

  For $a > 0$, $y = a(x - b)^2 + c$ has a minimum turning point at (b, c).

  For $a < 0$, $y = a(x - b)^2 + c$ has a maximum turning point at (b, c).

- *y-axis intercept*

  All points on the y-axis have an x-coordinate of zero. Hence substituting $x = 0$ into $y = a(x - b)^2 + c$ determines the y-axis intercept.

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EXAMPLE 3

For the graph of $y = 2(x - 2)^2 - 5$, determine

a the equation of the line of symmetry

b the coordinates of the maximum/minimum turning point, stating which of these it is

c the coordinates of the point where it cuts the y-axis

d Show these features on a sketch of the graph of $y = 2(x - 2)^2 - 5$.

Solution

a The quadratic function $y = a(x - b)^2 + c$ has line of symmetry $x = b$.

Hence $y = 2(x - 2)^2 - 5$ has line of symmetry $x = 2$.

b For $a > 0$, $y = a(x - b)^2 + c$ has a minimum turning point at (b, c).

Hence $y = 2(x - 2)^2 - 5$ has a minimum turning point at $(2, -5)$.

c All points on the y-axis have an x-coordinate of zero.

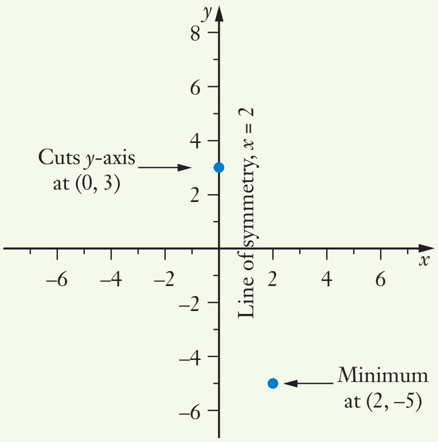
Given $\quad y = 2(x - 2)^2 - 5$,

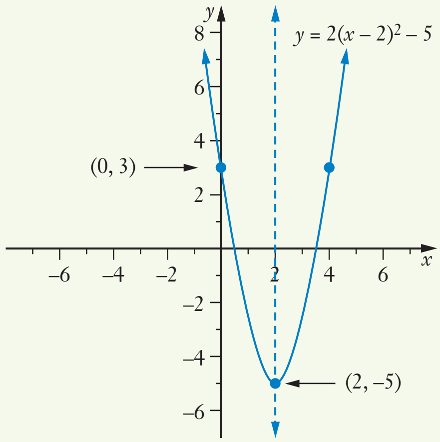
when $x = 0$, $\quad y = 2(0 - 2)^2 - 5$

$\quad = 3$.

The graph of $y = 2(x - 2)^2 - 5$ cuts the y-axis at $(0, 3)$.

d Placing the information from the previous parts on a graph, below left, the sketch can be completed, below right. (Note how the point symmetrical with (0, 3) has been included to help with the sketch.)





Note: When a question requires a *sketch* of a graph to be made, it does not mean neatness and reasonable accuracy can be ignored. We would not expect to be able to rely on great accuracy of values read from a sketch graph but the sketch should be neat, reasonably accurate and with the noteworthy features of the graph evident.

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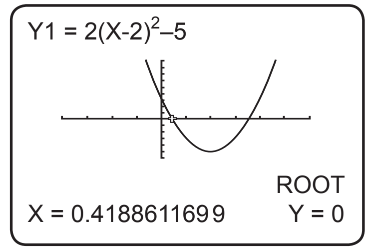
- The reader can confirm the correctness of the previous sketch by viewing the graph of

$$y = 2(x - 2)^2 - 5$$

on a graphic calculator.



- The calculator display shown also indicates that the graph cuts the x-axis at (0.4188611699, 0). Asked to determine the other x-axis intercept the calculator gives (3.58113883, 0).

  These values can also be obtained algebraically as follows.

  All points on the x-axis have a y-coordinate of zero. Thus for $y = 2(x - 2)^2 - 5$,

  when $y = 0$, $\quad 0 = 2(x - 2)^2 - 5$

  Add 5 to each side: $\quad 5 = 2(x - 2)^2$

  Divide each side by 2: $\quad 2.5 = (x - 2)^2$

  Hence $\quad x - 2 = \pm\sqrt{2.5}$

  $x = 2 + \sqrt{2.5}$ or $2 - \sqrt{2.5}$

  $= 3.58$ or 0.42 (2 decimal places).

  Notice that whilst these x-axis intercepts were not immediately obvious from the equation, a very acceptable sketch was possible without knowing them. Indeed approximate values for these intercepts could have been obtained from the sketch on the previous page.

EXAMPLE 4

For the graph of $y = -(x - 3)^2 + 5$ determine

a the equation of the line of symmetry

b the coordinates of the maximum/minimum turning point, stating which of these it is

c the coordinates of the point where $y = -(x - 3)^2 + 5$ cuts the y-axis.

d Show these features on a sketch of the graph of $y = -(x - 3)^2 + 5$.

Solution

a The quadratic function $y = a(x - b)^2 + c$ has line of symmetry $x = b$.

Hence $y = -(x - 3)^2 + 5$ has line of symmetry $x = 3$.

b For $a < 0$, $y = a(x - b)^2 + c$ has a maximum turning point at (b, c).

$y = -(x - 3)^2 + 5$ has $a = -1$ and so has a maximum turning point at (3, 5).

c All points on the y-axis have an x-coordinate of zero.

Given $\quad y = -(x - 3)^2 + 5$,

when $x = 0$, $\quad y = -(0 - 3)^2 + 5$

$= -4$.

The graph of $y = -(x - 3)^2 + 5$ cuts the y-axis at (0, –4).

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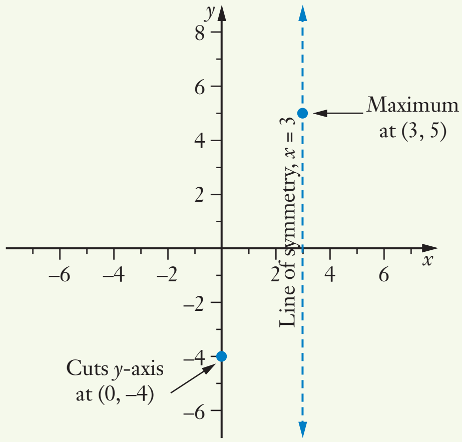
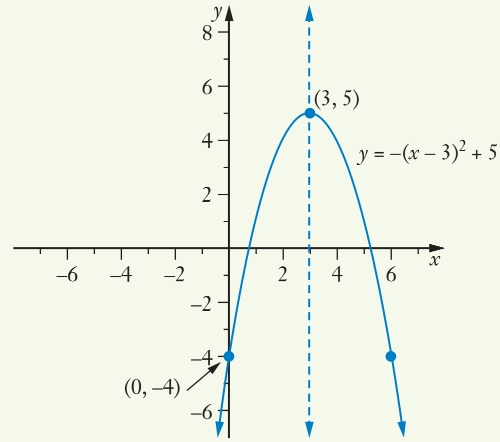
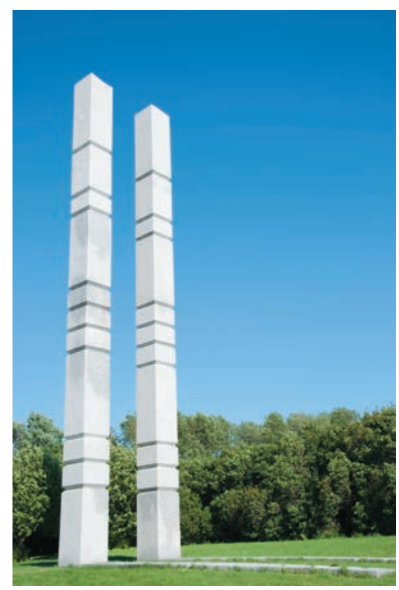
d Placing the information from the previous parts on a graph, below left, the sketch can be completed, below right. (Note the inclusion of the point symmetrical with (0, –4) to help with the sketch.)





Rule given in the form $y = a(x - b)(x - c)$



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- *y-axis intercept*

  All points on the y-axis have an x-coordinate of zero. Hence substituting $x = 0$ into the equation $y = a(x - b)(x - c)$ allows the y-axis intercept to be determined.

- *x-axis intercepts*

  If $x = b$, $\quad y = a(b - b)(b - c) = a(0)(b - c) = 0$ $\qquad$ If $x = c$, $\quad y = a(c - b)(c - c) = a(c - b)(0) = 0$

  But if a point has a y-coordinate of zero it must lie on the x-axis.

  Hence the graph of $y = a(x - b)(x - c)$ cuts the x-axis at $(b, 0)$ and at $(c, 0)$.

- *Line of symmetry*

  With the curve cutting the x-axis at $(b, 0)$ and $(c, 0)$ the line of symmetry must cut the x-axis at a point mid-way between these two points. This allows the equation of the line of symmetry to be determined.

- *Turning point*

  The turning point must lie on the line of symmetry and:

  for $a > 0$, $y = a(x - b)(x - c)$ will have a **minimum** turning point,

  for $a < 0$, $y = a(x - b)(x - c)$ will have a **maximum** turning point.

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EXAMPLE 5

For the graph of $y = (x + 1)(x - 5)$ determine

a the coordinates of the y-axis intercept

b the coordinates of the x-axis intercepts

c the equation of the line of symmetry

d the nature and location of the turning point.

e Sketch the curve $y = (x + 1)(x - 5)$.

Solution

a For every point on the y-axis, $x = 0$.

If $x = 0$,
$$\begin{aligned} y &= (0 + 1)(0 - 5) \\ &= (1)(-5) \\ &= -5 \end{aligned}$$

The curve cuts the y-axis at the point $(0, -5)$.

b For every point on the x-axis, $y = 0$.

This will occur when $x + 1 = 0$ and when $x - 5 = 0$,

i.e. when $x = -1$ and when $x = 5$.

The curve cuts the x-axis at the points $(-1, 0)$ and $(5, 0)$.

c The line of symmetry will cut the x-axis at the point mid-way between $(-1, 0)$ and $(5, 0)$, i.e. the point $(2, 0)$.

The line of symmetry is the line $x = 2$.

d $y = a(x - b)(x - c)$ has a *minimum* turning point when $a > 0$.

For $y = (x + 1)(x - 5)$, $a = 1$ so we have a minimum turning point.

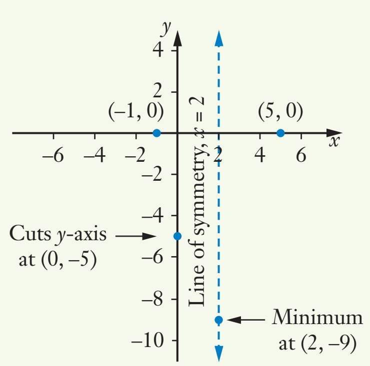
The turning point must lie on the line of symmetry.

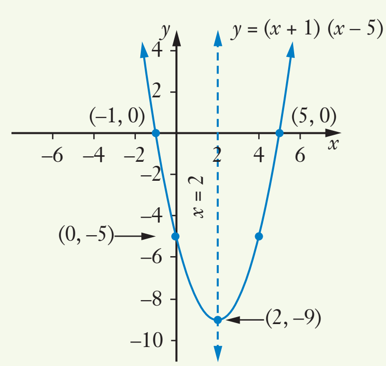
Hence its x-coordinate must equal 2.

If $x = 2$,
$$\begin{aligned} y &= (2 + 1)(2 - 5) \\ &= (3)(-3) \\ &= -9. \end{aligned}$$

The curve has a minimum turning point at $(2, -9)$.

e Placing the information from the previous parts on a graph, below left, the sketch can be completed, below right. (Note how the point symmetrical with $(0, -5)$ has been included to help with the sketch.)





Features of a parabola

Sketching quadratic functions

Rule given in the form $y = ax^2 + bx + c$

- *Line of symmetry*
  The graph of $y = ax^2 + bx + c$ has the line $x = -\frac{b}{2a}$ as its line of symmetry.
  This claim can be justified as follows:
  The graph of $y = ax^2 + bx + c$ is that of $y = ax^2 + bx$ moved up c units.
  Now $y = ax^2 + bx$ cuts the x-axis when $0 = ax^2 + bx$,
  i.e. $0 = x(ax + b)$.
  Hence the x-axis intercepts will be $x = 0$ and $x = -\frac{b}{a}$.
  The line of symmetry of $y = ax^2 + bx$, and hence of $y = ax^2 + bx + c$, will pass through the midpoint of these x-axis intercepts, which is $-\frac{b}{2a}$.
- *Turning point*
  The turning point must lie on the line of symmetry and:
  for $a > 0$, $y = ax^2 + bx + c$ this will be a **minimum** turning point,
  and for $a < 0$, $y = ax^2 + bx + c$ this will be a **maximum** turning point.
- *y-axis intercept*
  All points on the y-axis have an x-coordinate of zero. Substituting $x = 0$ into the equation $y = ax^2 + bx + c$ allows the y-axis intercept to be determined as $(0, c)$.

EXAMPLE 6

For the graph of $y = 2x^2 - 6x + 1$ determine

a the line of symmetry

b the nature and location of the turning point

c the y-axis intercept coordinates.

d Hence sketch the curve.

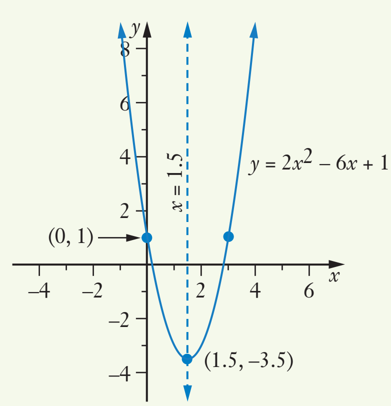
Solution

a The graph of $y = ax^2 + bx + c$ has the line $x = -\frac{b}{2a}$ as its line of symmetry.
$y = 2x^2 - 6x + 1$ has $a = 2$, $b = -6$ and $c = 1$.
The line of symmetry is $x = 1.5$.

b With the coefficient of x^2 being positive, we have a minimum turning point.
The turning point must lie on the line of symmetry. Thus $x = 1.5$.
With $x = 1.5$, $y = 2(1.5)^2 - 6(1.5) + 1$
$= -3.5$.
The turning point is a minimum and occurs at $(1.5, -3.5)$.

c Every point on the y-axis has an x-coordinate of zero.
With $x = 0$, $y = 2(0)^2 - 6(0) + 1$
$= 1$
The y-axis intercept has coordinates $(0, 1)$.

d The sketch can be completed, as shown on the right. (Note how the point symmetrical with $(0, 1)$ has been included to help with the sketch.)



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Exercise 5C

Rule given in the form $y = a(x - p)^2 + q$

1 For the graph of $y = (x + 1)^2 - 4$ determine

- **a** the equation of the line of symmetry
- **b** the coordinates of the maximum/minimum turning point, stating which of these it is
- **c** the coordinates of the point where $y = (x + 1)^2 - 4$ cuts the y-axis.
- **d** Show these features on a sketch of $y = (x + 1)^2 - 4$.

2 For the graph of $y = (x - 3)^2 + 5$ determine

- **a** the equation of the line of symmetry
- **b** the coordinates of the maximum/minimum turning point, stating which of these it is
- **c** the coordinates of the point where $y = (x - 3)^2 + 5$ cuts the y-axis.
- **d** Show these features on a sketch of $y = (x - 3)^2 + 5$.

3 For the graph of $y = -2(x - 1)^2 + 3$ determine

- **a** the equation of the line of symmetry
- **b** the coordinates of the maximum/minimum turning point, stating which of these it is
- **c** the coordinates of the point where $y = -2(x - 1)^2 + 3$ cuts the y-axis.
- **d** Show these features on a sketch of $y = -2(x - 1)^2 + 3$.

Rule given in the form $y = a(x - b)(x - c)$

4 For the graph of $y = (x - 3)(x - 7)$ determine

- **a** the coordinates of the point where the curve cuts the y-axis
- **b** the coordinates of the points where the curve cuts the x-axis
- **c** the equation of the line of symmetry
- **d** the nature and location of the turning point.
- **e** Show these features on a sketch of $y = (x - 3)(x - 7)$.

5 For the graph of $y = (x - 3)(x + 4)$ determine

- **a** the coordinates of the point where the curve cuts the y-axis
- **b** the coordinates of the points where the curve cuts the x-axis
- **c** the equation of the line of symmetry
- **d** the nature and location of the turning point.
- **e** Show these features on a sketch of $y = (x - 3)(x + 4)$.

6 For the graph of $y = (x + 2)(x + 4)$ determine

- **a** the coordinates of the point where the curve cuts the y-axis
- **b** the coordinates of the points where the curve cuts the x-axis
- **c** the equation of the line of symmetry
- **d** the nature and location of the turning point.
- **e** Show these features on a sketch of $y = (x + 2)(x + 4)$.

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Rule given in the form $y = ax^2 + bx + c$


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7 For the graph of $y = x^2 + 4x - 12$ determine

a the equation of the line of symmetry

b the location and nature of the turning point

c the coordinates of the y-axis intercept.

d Hence sketch the curve.

8 For the graph of $y = x^2 - 6x + 1$ determine

a the equation of the line of symmetry

b the location and nature of the turning point

c the coordinates of the y-axis intercept.

d Hence sketch the curve.

9 For the graph of $y = -2x^2 + 4x + 1$ determine

a the equation of the line of symmetry

b the location and nature of the turning point

c the coordinates of the y-axis intercept.

d Hence sketch the curve.

10 For the graph of $y = 8x - 2x^2 - 3$ determine

a the equation of the line of symmetry

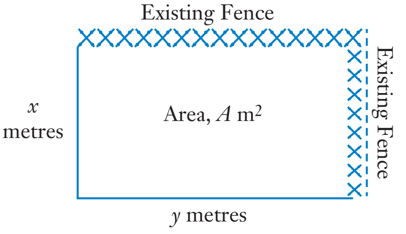
b the location and nature of the turning point

c the coordinates of the y-axis intercept.

d Hence sketch the curve.

Applications

11 A dog owner wishes to enclose a rectangular area in his backyard for the dog. The owner wishes to use existing fencing along two adjacent sides of the rectangle and has 14 metres of new fencing available for the other two sides. Suppose that we let the dimensions of the rectangle be x metres and y metres and the area be A m^2 as shown.



a Explain why $x + y = 14$.

b Explain why $A = x(14 - x)$.

c With A on the vertical axis and x on the horizontal axis make a sketch of

$$A = x(14 - x).$$

d What is the greatest rectangular area the owner can enclose and what would be the dimensions that give this greatest area?

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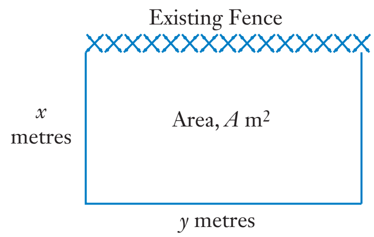
12 A gardener has 20 metres of fencing and wishes to fence off a rectangular area using the twenty metres of fencing on three sides and an existing fence on the fourth. Suppose we define the variables x, y and A as shown in the diagram.



a Explain why $A = 20x - 2x^2$.

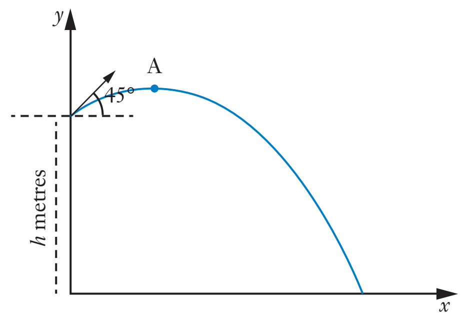
b With A on the vertical axis and x on the horizontal axis make a sketch of

$$A = 20x - 2x^2.$$

c What is the greatest rectangular area the gardener can enclose and what would be the dimensions that give this greatest area?

For the following questions view the graph on your calculator if you wish.

13 The diagram below shows the motion of a particle fired with initial speed of 7 m/s at 45° to the horizontal, from a position that is h metres above ground level.



The equation of the path of the particle is

$$y = -0.2(x - 2.5)^2 + 11.25$$

with x and y axes as shown in the diagram, and 1 metre to 1 unit on each axis. Find

a the coordinates of A, the highest point on the path of the particle

b the value of h.

c State whether the graph shown is concave up or concave down.

14 A housing market analyst was trying to predict the best time to purchase a house during a slump in sales and the consequent fall in prices. She felt that the decline in prices and the rise that was expected to follow would be according to the rule:

$$\text{Average house price (in \$1000s)} = 0.6t^2 - 12t + 590$$

where t is the time in months.

If her quadratic model of the situation is correct

a what is the average house price when $t = 0$?

b what is the average house price when $t = 15$?

c when (i.e for what value of t) should she purchase a house and what should she expect the average price to be then?

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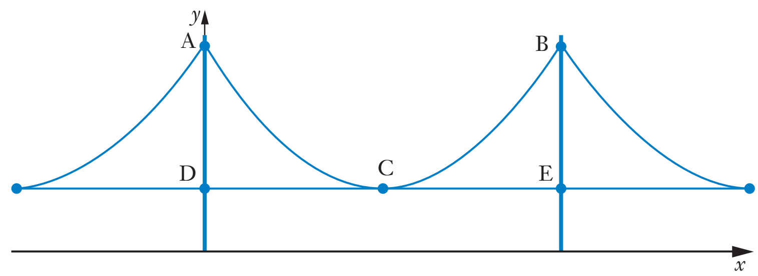
15 The height above ground, h metres, of a stone projected vertically upwards, from ground level with initial speed 49 m/s, is given by $h = 49t - 4.9t^2$, where t is the time in seconds after projection. Determine the maximum value of h and the value of t for which it occurs.

16



The diagram above shows a symmetrical suspension bridge over a river. The x-axis is the water level, the y-axis is as indicated, 1 metre is 1 unit on each axis and the support wire from A to B has the equation $y = \frac{3}{160}(x - 40)^2 + 15$.

a Is the curve from A to B concave up or concave down?

b How high is the bridge above the water level (i.e. how high is point C above the x-axis)?

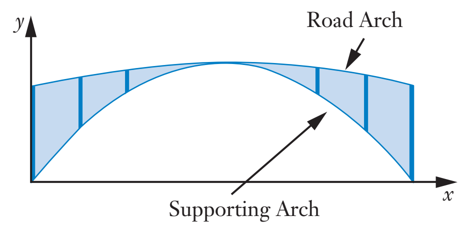
c Find the equation of the line of symmetry of the quadratic curve ACB.

How far is it from

d D to C? **e** D to E? **f** D to A?

17 A road bridge is to be constructed over a tidal river. The road arch is in the shape of a quadratic function, as is the supporting arch (see diagram).

With x- and y- axes as shown the equations of each arch are as follows.



(1 metre to 1 unit on each axis.)

Road arch: $y = -\frac{x^2}{2250} + \frac{2x}{15} + 40$

Supporting arch: $y = \frac{2x}{3} - \frac{x^2}{450}$

The x-axis is the mean water level with the high and low tide levels being four metres either side of this mean.

a Are the road arch and supporting arch concave up or concave down?

b Calculate the value of x at the mid-point of the bridge.

c Calculate the length of the vertical strut between the support arch and the road arch at a point one quarter of the way along the bridge.

d Calculate the maximum clearance between the water and the bridge at

i low tide **ii** high tide.

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Tables of values

As the Preliminary section mentioned, quadratic functions have tables of values with a *constant second difference pattern*.

For example, for $y = 2x^2 - 6x + 1$:

x	0	1	2	3	4	5	6
y	1	–3	–3	1	9	21	37

1st difference: –4, 0, 4, 8, 12, 16

2nd difference: 4, 4, 4, 4, 4

This constant second difference becomes clear when we consider a table of values for the general quadratic $y = ax^2 + bx + c$:

x	0	1	2	3	4
$y = ax^2 + bx + c$	c	$a + b + c$	$4a + 2b + c$	$9a + 3b + c$	$16a + 4b + c$

1st difference: $a + b$, $3a + b$, $5a + b$, $7a + b$

2nd difference: $2a$, $2a$, $2a$

Consider the following table of values with, 1st and 2nd difference patterns as shown:

x	0	1	2	3	4
y	–4	1	10	23	40

1st difference: 5, 9, 13, 17

2nd difference: 4, 4, 4

- The constant second difference pattern confirms that the relationship is quadratic.
- Comparing the constant second difference of 4 with $2a$, the constant second difference pattern in the table for the general quadratic, $y = ax^2 + bx + c$, gives us that $a = 2$.
- From the table for the general quadratic we see that when $x = 0$, $y = c$. Thus $c = -4$.
- Comparing the y values for $x = 1$: $a + b + c = 1$

  Thus, with $a = 2$ and $c = -4$, $b = 3$.

The table of values is for the quadratic $y = 2x^2 + 3x - 4$.

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Consider the table and difference patterns shown below:

x	–2	–1	0	1	2	3	4
y	34	19	10	7	10	19	34

1st difference   – 15   –9   –3   3   9   15

2nd difference   6   6   6   6   6

The constant second difference pattern confirms that the table is for a quadratic function.

To determine the equation of the function we could compare the given table with that of the table for $y = ax^2 + bx + c$, as we did on the previous page, to determine that

$$2a = 6, \qquad c = 10, \qquad a + b + c = 7.$$

Hence $\quad a = 3, \qquad c = 10, \qquad b = –6.$

The rule is $y = 3x^2 – 6x + 10$.

Alternatively, from the symmetrical nature of the y values, the line of symmetry is $x = 1$ and the minimum point has coordinates (1, 7). Thus the rule is of the form

$$y = a(x – 1)^2 + 7.$$

The point (0, 10) must ‘fit’. Thus $\quad 10 = a(0 – 1)^2 + 7,$

giving $\quad a = 3.$

The rule is $\quad y = 3(x – 1)^2 + 7.$

The reader should confirm that these answers, $y = 3x^2 – 6x + 10$

and $y = 3(x – 1)^2 + 7$, are indeed the same.

Exercise 5D

For each of the tables shown in questions 1 to 12, by considering difference patterns, determine whether the relationship between x and y is linear, quadratic or neither of these. For those relationships that are either linear or quadratic determine the rule.

1

x	0	1	2	3	4	5
y	5	12	21	32	45	60

2

x	0	1	2	3	4	5
y	0	1	8	27	64	125

3

x	0	1	2	3	4	5
y	3	5	9	15	23	33

4

x	0	1	2	3	4	5
y	1	6	11	16	21	26

5

x	0	1	2	3	4	5
y	2	3	6	11	18	27

6

x	0	1	2	3	4	5
y	π	2π	3π	4π	5π	6π

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7

x	0	1	2	3	4	5
y	3	6	12	24	48	96

8

x	4	3	0	5	1	2
y	40	28	4	54	10	18

9

x	1	4	2	0	5	3
y	11	35	19	3	43	27

10

x	1	3	2	5	4	0
y	5	21	11	53	35	3

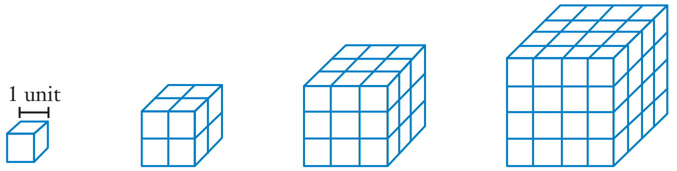
11

x	–1	0	1	2	3	4	5
y	28	13	4	1	4	13	28

12

x	–2	0	2	4	6	8
y	–20	–4	4	4	–4	–20

13



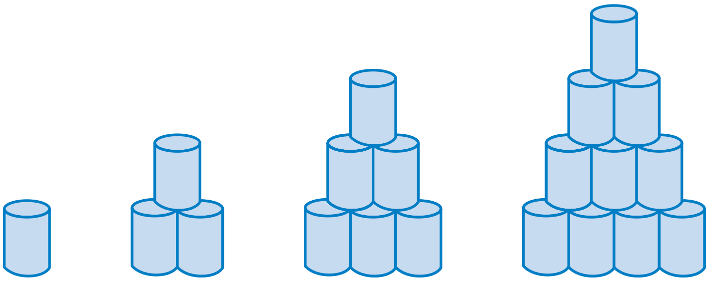
a Copy and complete the table shown below for the pattern shown above.

Length of side of cube (L units)	1	2	3	4	5	6
Surface area of cube (n units^2)	6	24				

b Use difference patterns to determine whether the relationship between L and n is linear, quadratic or neither of these.

c If linear or quadratic, determine the rule.

14



a Copy and complete the table shown below for the pattern shown above.

Number of rows of cans (r)	1	2	3	4	5	6
Number of cans (n)	1	3	6			

b Use difference patterns to determine whether the relationship between r and n is linear, quadratic or neither of these.

c If linear or quadratic, determine the rule.

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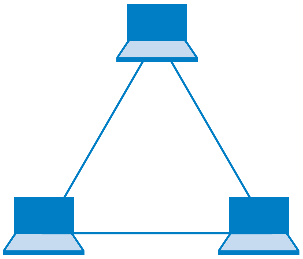
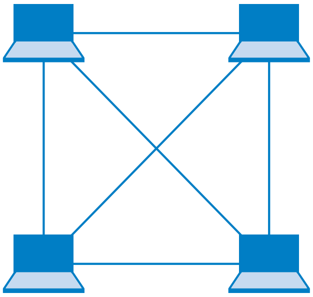
Social networks

Two computers in a network can involve 1 connection.

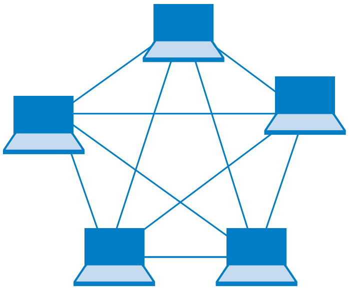
Three computers in a network can involve 3 connections.



Four computers in a network can involve 6 connections.



Five computers in a network can involve 10 connections.



How many connections can be involved in a six computer network?

How many connections can be involved in a seven computer network?

Investigate.

Projectiles

What has the path of a projectile got to do with parabolas and quadratic functions?

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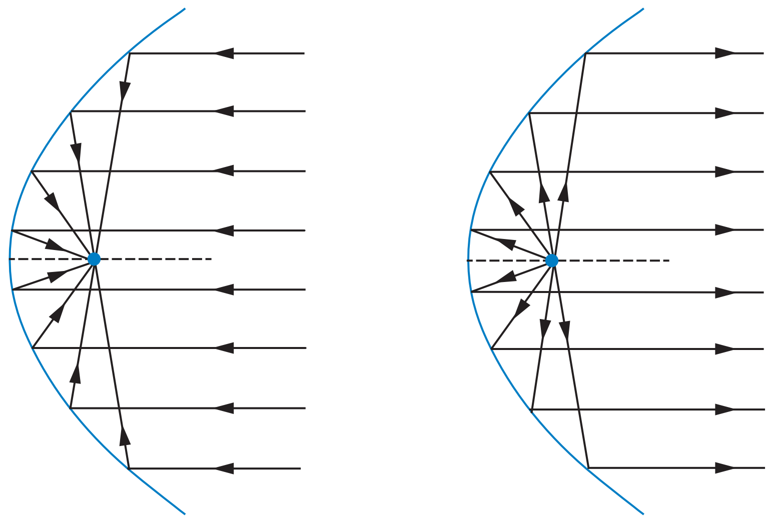
Transmitters and receiving dishes

Why is the parabolic shape of significance for car headlights, receiving dishes and transmitters?



Stopping distances

Suppose that a vehicle travelling at a particular speed requires y metres to come to rest once normal braking is applied.

Under the same conditions, if the vehicle is travelling twice as fast as previously it will require not $2y$ but $4y$ metres to come to rest once normal braking is applied. I.e. the braking distance required and the distance travelled are *not* linearly related – double the speed, quadruple the distance required.

When something occurs ahead of a moving vehicle, requiring the driver to apply the brakes and come to a stop, the vehicle will initially travel on whilst the driver reacts to the situation, and then the vehicle will travel further 'under braking', until it is brought to rest.

Suppose that for a particular vehicle and road conditions the reaction distance, braking distance and total stopping distance at various speeds are as follows:

Speed (x km/h)	40	50	60	70	80	90	100	110
Reaction distance (R m)	12	15	18	21	24	27	30	33
Braking distance (B m)	16	25	36	49	64	81	100	121
Total stopping distance (T m)	28	40	54	70	88	108	130	154

What functions and rules are involved here?

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Completing the square

We saw earlier in this chapter that when a quadratic function is given in the form

$$y = a(x - p)^2 + q,$$

it is an easy matter to determine the line of symmetry, $x = p$, and the coordinates of the turning point, (p, q).

For example the graph of the quadratic function

$$y = 2(x - 5)^2 - 7$$

has $x = 5$ as its line of symmetry and has a turning point (a minimum) at $(5, -7)$.

Given $\qquad y = 2(x - 5)^2 - 7$

and expanding: $\qquad y = 2(x - 5)(x - 5) - 7$

$$= 2(x^2 - 10x + 25) - 7$$

$$= 2x^2 - 20x + 43$$

we obtain the rule in its $y = ax^2 + bx + c$ form.

$$y = 2x^2 - 20x + 43$$

However, how would we go about changing a quadratic rule given in the form

$$y = ax^2 + bx + c$$

to the form $\qquad y = a(x - p)^2 + q$?

The answer is that we use the technique referred to as 'completing the square'.

To use this method we need to be familiar with the expansion:

$$(x \pm e)^2 = x^2 \pm 2ex + e^2.$$

Note especially that the number on the end, e^2, is *the square of half the coefficient of x*.

EXAMPLE 7

Express $y = x^2 + 3x - 5$ in the form $y = a(x - p)^2 + q$.

Solution

Given $\qquad y = x^2 + 3x - 5,$

First create a gap to allow 'the square of half the coefficient of x' to be inserted.

$$y = x^2 + 3x \qquad - 5$$

Then add, and then subtract, $\left(\frac{3}{2}\right)^2$, which is *the square of half the coefficient of x*.

$$y = x^2 + 3x + \left(\frac{3}{2}\right)^2 - 5 - \left(\frac{3}{2}\right)^2$$

$\therefore \qquad y = \left(x + \frac{3}{2}\right)^2 - \frac{29}{4}$

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EXAMPLE 8

Express

a $y = x^2 + 6x - 3$ **b** $y = 2x^2 - 4x + 1$

in the form $y = a(x - p)^2 + q$, and hence determine the nature and location of the turning point of the graph of the function.

Solution

a

$$\begin{aligned} y &= x^2 + 6x - 3 \\ &= x^2 + 6x \quad - 3 \\ &= x^2 + 6x + 9 - 3 - 9 \\ \therefore \quad y &= (x + 3)^2 - 12 \end{aligned}$$

Minimum turning point at (–3, –12).

b

$$\begin{aligned} y &= 2x^2 - 4x + 1 \\ &= 2(x^2 - 2x) \quad + 1 \\ &= 2(x^2 - 2x + 1 - 1) + 1 \\ &= 2(x^2 - 2x + 1) - 2 + 1 \\ \therefore \quad y &= 2(x - 1)^2 - 1 \end{aligned}$$

Minimum turning point at (1, –1).

Exercise 5E

Express each of the following in the form $y = a(x - p)^2 + q$ and hence determine the nature and location of the turning point of the graph of each function.

1 $y = x^2 + 4x - 1$ **2** $y = x^2 - 6x + 2$ **3** $y = x^2 - 8x + 10$

4 $y = x^2 + 6x + 3$ **5** $y = x^2 - 3x + 2$ **6** $y = x^2 - 5x + 3$

7 $y = -x^2 + 10x - 1$ **8** $y = 2x^2 - 12x + 3$ **9** $y = -2x^2 + 8x + 4$

10 $y = 2x^2 + 5x + 4$

If we were to be repeatedly carrying out a procedure like completing the square we could carry out the process on the general quadratic $y = ax^2 + bx + c$ in order to obtain a formula. Then, given a specific quadratic, we simply substitute the appropriate values of a, b and c.

$$\begin{aligned} y &= ax^2 + bx + c \\ &= a\left(x^2 + \frac{b}{a}x\right) + c \\ &= a\left(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} - \frac{b^2}{4a^2}\right) + c \\ &= a\left(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2}\right) - \frac{b^2}{4a} + c \\ &= a\left(x + \frac{b}{2a}\right)^2 + c - \frac{b^2}{4a} \end{aligned}$$

Thus the expression $y = ax^2 + bx + c$ can be written as $y = a\left(x + \frac{b}{2a}\right)^2 + c - \frac{b^2}{4a}$

The reader should now repeat Example 8 above by substituting appropriate values for a, b and c into the formula above and check that the answers obtained are the same as those given by the working at the top of the page.

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Miscellaneous exercise five

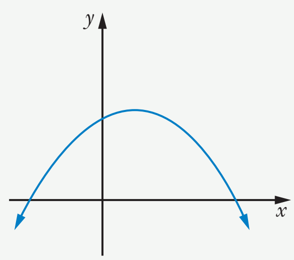
This miscellaneous exercise may include questions involving the work of this chapter, the work of any previous chapters, and the ideas mentioned in the Preliminary work section at the beginning of the book.

1 If $f(x) = 5x + 1$ and $g(x) = x^2 - 3$ determine

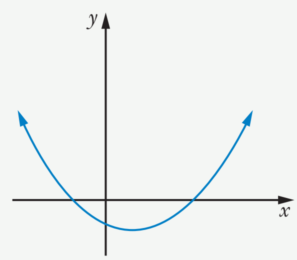
a $f(6)$ **b** $g(2)$ **c** $f(2) + g(6)$.

2 Classify each of the curves shown below as either 'concave up' or as 'concave down'.

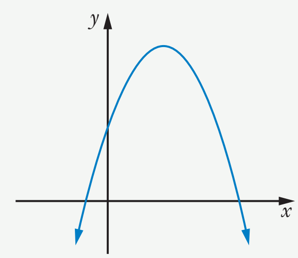
a



b



c



3 Given that all of the points A to F given below lie on the line $y = 2x - 5$, determine the values of a, b, c, d, e and f.

A$(3, a)$, B$(2, b)$, C$(-4, c)$, D$(2.5, d)$, E$(e, 13)$, F$(f, -5)$.

4 **a** Find the gradient of a line that is perpendicular to a line that has a gradient of 2.

b Find the gradient of a line that is perpendicular to $y = 3x - 4$.

c Find the gradient of a line that is perpendicular to $y = -0.2x + 1$.

d Find the equation of a line that passes through the point (3, 13) and is perpendicular to $y = -0.5x + 1$.

5 For the graph of $y = (x - 1)(x - 3)$, determine

a the coordinates of the y-axis intercept

b the coordinates of the x-axis intercepts

c the equation of the line of symmetry

d the nature and location of the turning point.

6 Copy and complete the following table.

Equation	Cuts y-axis	Line of symmetry	Turning point	
			Coordinates	Max or min?
$y = x^2 + 4x + 1$	(?, ?)	$x = ?$	(?, ?)	
$y = x^2 - 2x - 1$	(?, ?)	$x = ?$	(?, ?)	
$y = 2x^2 + 4x - 3$	(?, ?)	$x = ?$	(?, ?)	
$y = 2x^2 + 6x - 1$	(?, ?)	$x = ?$	(?, ?)	

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7 For the graph of $y = 2(x + 3)^2 - 4$ determine

a the equation of the line of symmetry

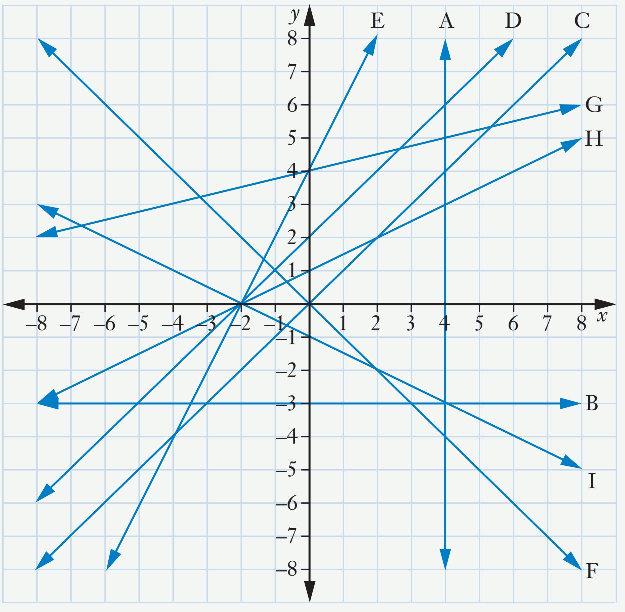
b the coordinates of the turning point.

The graph of $y = 2(x + 3)^2 - 4$ is moved 2 units to the right and 3 units up.

c What will be the equation of the line of symmetry of the 'new' curve?

d What will be the coordinates of the turning point of the 'new' curve?

8 Determine the equation of each of the straight lines A to I shown below.



9 The four curves I, II, III and IV shown on the right have their equations amongst the six listed below.

Select the appropriate equation for each of the curves.

$y = (x - 2)(x + 2)$

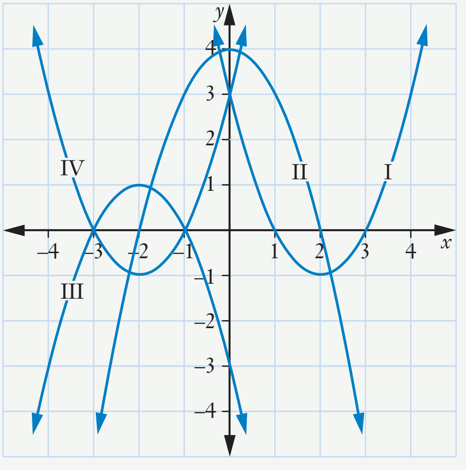
$y = (x - 1)(x - 3)$

$y = (x + 1)(x + 3)$

$y = -(x + 1)(x + 3)$

$y = (x + 2)(2 - x)$

$y = (x - 1)(x + 3)$



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10 Given that each of the following tables show linear relationships between x and y find the rule for each table and complete each table.

a

x	1	2	3	4	5	6	7	8
y				16	19			

b

x	1	2	3	4	5	6	7	8
y							13	15

c

x	1	2	3	4	5	6	7	8
y			11	9				

d

x	1	2	3	4	5	6	7	8
y		9		19				

e

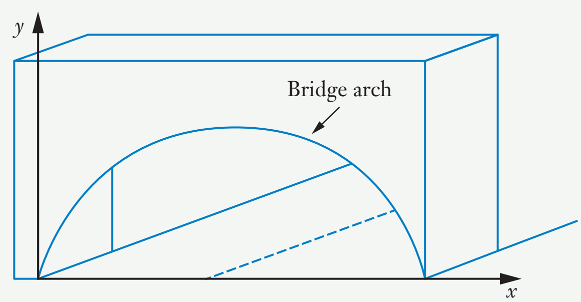
x	3	8	1	6	7	4	5	2
y			1				13	

11 Find the equation of the quadratic function that has exactly the same shape as that of $y = 3x^2$, has the same line of symmetry as $y = (x - 2)^2 + 3$ and that cuts the y-axis at the point (0, 15).

12 The diagram on the right shows a road under a bridge.

With x- and y-axes as shown the bridge arch (see diagram) has equation

$$y = \frac{5x}{16}(8 - x).$$



Determine

a the width of the road

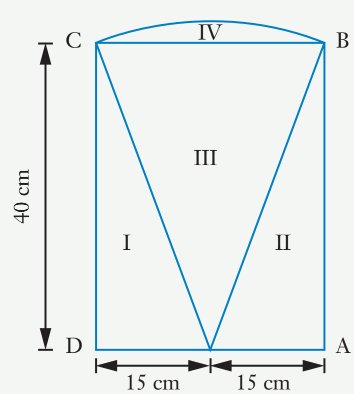
b the clearance at the centre of the bridge.

For a vehicle of width 2.3 metres determine the maximum height of the vehicle (in whole centimetres) if it is to pass under the bridge and

c only use its own side of the road

d use both carriageways.

13 A stained glass window is to be constructed to the dimensions shown on the right. The arc BC is part of a circle which has the midpoint of AD as its centre and ABCD is rectangular. The window consists of four pieces of glass, I, II, III and IV separated and surrounded by strips of lead (shown as the straight and curved lines in the diagram).



Find

a the area of each piece of glass to the nearest square centimetre, (ignore the thickness of the lead strips),

b the total length of all the lead strips, to the nearest centimetre.

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